Chapter 7

T. H. Pulliam

NASA Ames

Stability of Linear Systems

- Stability will be defined in terms of ODE's and $O\Delta E$'s
  - ODE: Couples System

$$\frac{d\vec{u}}{dt} = A\,\vec{u} - \vec{f}(t) \tag{1}$$

  - $O\Delta E$: Matrix form from applying Eq. 1

$$\vec{u}_{n+1} = C\,\vec{u}_n - \vec{g}_n \tag{2}$$

  - For Example: Euler Explicit $C = [I - hA]$

Inherent Stability of ODE's

- Stability of Eq. 1 depends entirely on the eigensystem of A.
- λ_m-spectrum of A: function of finite-difference scheme,BC

$$\boxed{\begin{array}{l}\text{For a } \textit{stationary} \text{ matrix } A\text{, Eq. 1 is } \textit{inherently stable} \text{ if,}\\ \text{when } \vec{f} \text{ is constant, } \vec{u} \text{ remains bounded as } t \to \infty.\end{array}} \tag{3}$$

- Note that inherent stability depends only on the transient solution of the ODE's.

$$\begin{aligned} \vec{u}(t) &= c_1 \left(e^{\lambda_1 h}\right)^n \vec{x}_1 + \cdots + c_m \left(e^{\lambda_m h}\right)^n \vec{x}_m + \cdots \\ &+ c_M \left(e^{\lambda_M h}\right)^n \vec{x}_M + P.S. \end{aligned} \tag{4}$$

- ODE's are inherently stable if and only if

$$\boxed{\Re(\lambda_m) \leq 0 \quad \text{for all} \quad m} \tag{5}$$

- For inherent stability, all of the λ eigenvalues must lie on, or to the left of, the imaginary axis in the complex λ plane.
- This criterion is satisfied for the model ODE's representing both diffusion and biconvection.

Numerical Stability of $O\Delta E$'s

- Stability of Eq. 2 related to the eigensystem of its matrix, C.
- σ_m-spectrum of C: determined by the $O\Delta E$ and are a function of λ_m

$$\begin{aligned} \vec{u}_n &= c_1(\sigma_1)^n\,\vec{x}_1 + \cdots + c_m(\sigma_m)^n\,\vec{x}_m + \cdots \\ &+ c_M(\sigma_M)^n\,\vec{x}_M \;+\; P.S. \end{aligned} \tag{6}$$

- Spurious roots play a similar role in stability.
- The $O\Delta E$ companion to Statement 3 is

$$\boxed{\begin{array}{l}\text{For a } \textit{stationary} \text{ matrix } C\text{, Eq. 2 is } \textit{numerically stable} \\ \text{if, when } \vec{g} \text{ is constant, } \vec{u}_n \text{ remains bounded as } n \to \infty.\end{array}} \tag{7}$$

- Definition of stability: referred to as asymptotic or time stability.
- Time-marching method is numerically stable if and only if

$$\boxed{|(\sigma_m)_k| \le 1 \quad \text{for all } m \text{ and } k} \tag{8}$$

- This condition states that, for numerical stability, all of the σ eigenvalues (both principal and spurious, if there are any) must lie on or inside the unit circle in the complex σ-plane.
- This definition of stability for OΔE's is consistent with the stability definition for ODE's.

Review

- Our Approach leads to
  - The PDE's are converted to ODE's by approximating the space derivatives on a finite mesh.
  - Inherent stability of the ODE's is established by guaranteeing that $\Re(\lambda) \leq 0$.
  - Time-march methods are developed which guarantee that $|\sigma(\lambda h)| \leq 1$ and this is taken to be the condition for numerical stability.

Time-Space Stability and Convergence of OΔE's

- A more classical view (but consistent) in the time-space sense.
  - The homogeneous part of Eq. 2, $\vec{u}_{n+1} = C\vec{u}_n$
  - Applying simple recursion $\vec{u}_n = C^n\vec{u}_0$
  - Using vector and matrix p-norms

$$||\vec{u}_n|| = ||\mathrm{C}^n\vec{u}_0|| \le ||\mathrm{C}^n|| \cdot ||\vec{u}_0|| \le ||\mathrm{C}||^n \cdot ||\vec{u}_0|| \tag{9}$$

  - Assume that the initial data vector is bounded, the solution vector is bounded if

$$||\mathrm{C}|| \le 1 \tag{10}$$

where $||\mathrm{C}||$ represents any p-norm of C.

- This is often used as a *sufficient* condition for stability.
- Well known relation between spectral radii and matrix norms
  * The spectral radius of a matrix is its L_2 norm when the matrix is normal, i.e., it commutes with its transpose.
  * The spectral radius is the *lower bound* of all norms.

- The matrix norm approach and the $\sigma - \lambda$ analysis are consistent when both A and C have a complete eigensystem.

Numerical Stability Concepts: Complex σ-Plane

- σ-Root Traces Relative to the Unit Circle
- The $O\Delta E$ solution to the homogeneous part

$$\vec{u}_n = c_1 \sigma_1^n \vec{x}_1 + \cdots + c_m \sigma_m^n \vec{x}_m + \cdots + c_M \sigma_M^n \vec{x}_M$$

- Semi-discrete approach leads to a relation between the σ and the λ eigenvalues.
- Numerical stability of the $O\Delta E$ requires that σ-roots lie within unit circle in the complex σ-plane.
- Trace the locus of the σ-roots as a function of the parameter λh

Stability in the Complex-σ Plane

- Define $\sigma_{exact} = e^{\lambda h}$ and Separate:
  - Dissipation $(\lambda h = \beta < 0)$ —— Convection $(\lambda h = i\omega)$
- Plot $Real(\sigma)$ and $Imag(\sigma)$ for varying λh of both types.

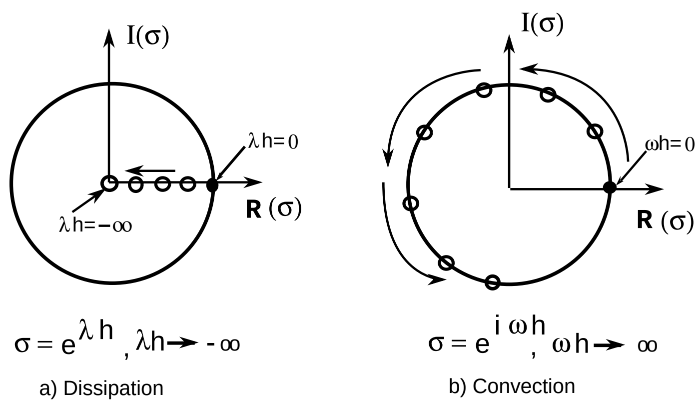


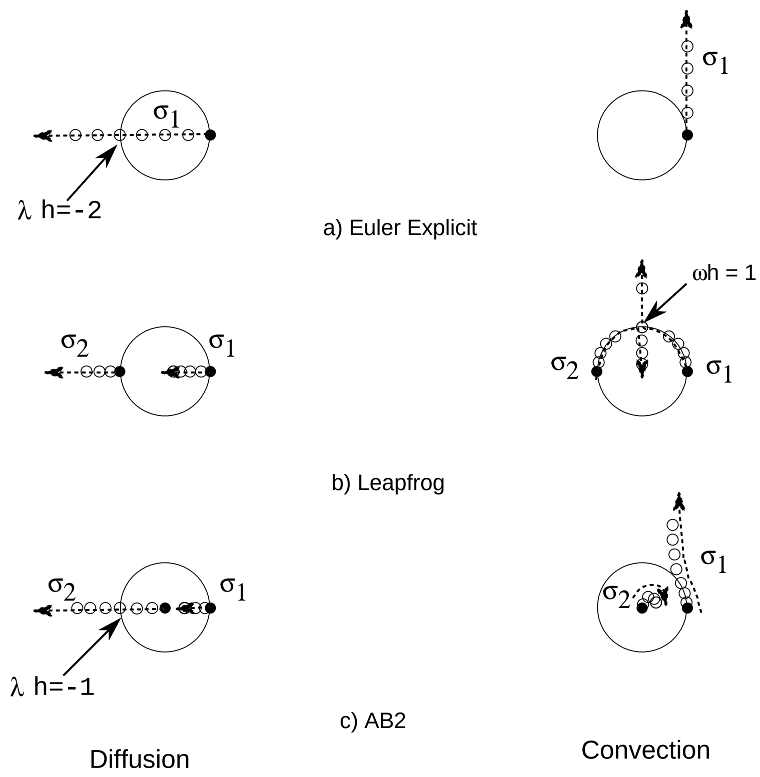
Figure 1: Exact traces of σ-roots for model equations.

$\sigma - \lambda$ Relations for Various Schemes

1.	$\sigma - 1 - \lambda h = 0$	Explicit Euler
2.	$\sigma^2 - 2\lambda h \sigma - 1 = 0$	Leapfrog
3.	$\sigma^2 - (1 + \frac{3}{2}\lambda h)\sigma + \frac{1}{2}\lambda h = 0$	AB2
4.	$\sigma^3 - (1 + \frac{23}{12}\lambda h)\sigma^2 + \frac{16}{12}\lambda h \sigma - \frac{5}{12}\lambda h = 0$	AB3
5.	$\sigma^(1 - \lambda h) - 1 = 0$	Implicit Euler
6.	$\sigma(1 - \frac{1}{2}\lambda h) - (1 + \frac{1}{2}\lambda h) = 0$	Trapezoidal
7.	$\sigma^2(1 - \frac{2}{3}\lambda h) - \frac{4}{3}\sigma + \frac{1}{3} = 0$	2nd-Order Backward
8.	$\sigma^2(1 - \frac{5}{12}\lambda h) - (1 + \frac{8}{12}\lambda h)\sigma + \frac{1}{12}\lambda h = 0$	AM3
9.	$\sigma^2 - (1 + \frac{13}{12}\lambda h + \frac{15}{24}\lambda^2 h^2)\sigma + \frac{1}{12}\lambda h(1 + \frac{5}{2}\lambda h) = 0$	ABM3
10.	$\sigma^3 - (1 + 2\lambda h)\sigma^2 + \frac{3}{2}\lambda h \sigma - \frac{1}{2}\lambda h = 0$	Gazdag
11.	$\sigma - 1 - \lambda h - \frac{1}{2}\lambda^2 h^2 = 0$	RK2
12.	$\sigma - 1 - \lambda h - \frac{1}{2}\lambda^2 h^2 - \frac{1}{6}\lambda^3 h^3 - \frac{1}{24}\lambda^4 h^4 = 0$	RK4
13.	$\sigma^2(1 - \frac{1}{3}\lambda h) - \frac{4}{3}\lambda h \sigma - (1 + \frac{1}{3}\lambda h) = 0$	Milne 4th

Table 7.1. Some $\lambda - \sigma$ Relations

Traces of σ-roots for various methods.



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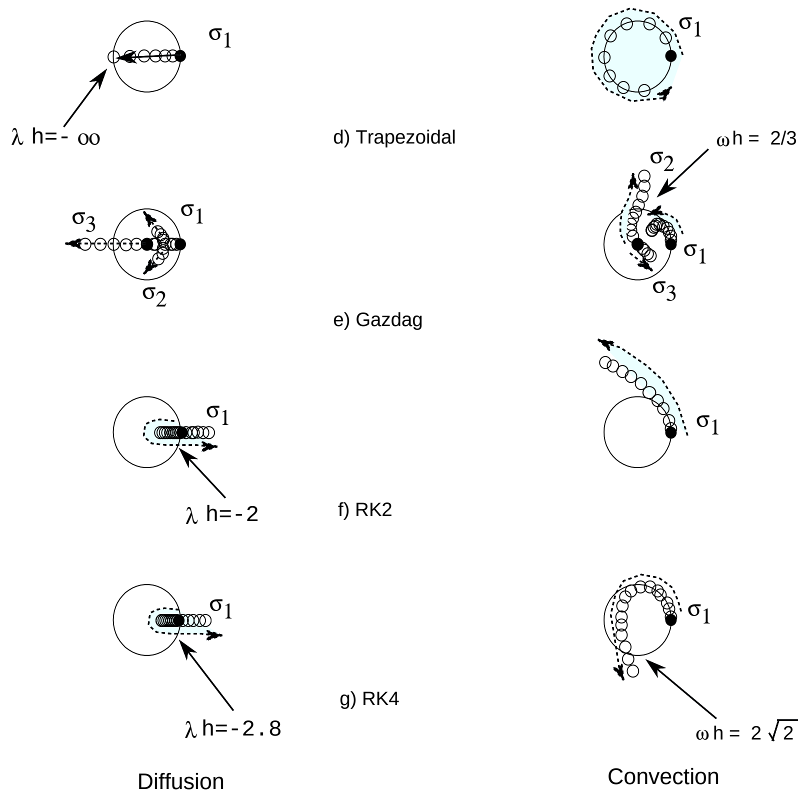


Types of Stability

- Conditional Stability: Explicit Methods
  - $O\Delta E$'s where $\lambda h \leq Constant$
  - λ spectrum, e.g. $\lambda_b h = -\frac{ah}{\Delta x}(1 - cos(k\Delta x) + isin(k\Delta x))$
  - Given Δx, wave speed a, and difference scheme: λ fixed
  - Adjust $h = \Delta t$ to satisfy stability bound
  - Time accuracy: use an appropriate h
  - Mildly-unstable: Prof. Milton VanDyke

    Lock bike fork and peddle as fast as you can, you may cross the street before you fall over and a truck hits you.

- Un-Conditional Stability: Implicit Methods

> A numerical method is *unconditionally stable* if it is stable for all ODE's that are inherently stable.

  - $O\Delta E$'s where $\lambda h \to \infty$ is stable
  - Time accuracy: use an appropriate h
  - Steady-State: any h which converges fast.
  - Computationally expensive compared with Explicit Methods

Stability Contours in the Complex λh Plane.

- Another view of stability properties of a time-marching method is to plot the locus of the complex λh for which $|\sigma| = 1$
- $|\sigma|$ refers to the maximum absolute value of any σ, principal or spurious, that is a root to the characteristic polynomial for a given λh.
- Inherently stable ODE's lies in the left half complex-sigma plane

Example for Euler Explicit

- Euler explicit: $\sigma_{ee} = 1 + h\lambda$
  - Wave equation: central differencing, $\lambda_c = -ai\frac{sin(k\Delta x)}{\Delta x}$

$$\sigma_{ee} = 1 - \frac{ah}{\Delta x} isin(k\Delta x)$$

  - $|\sigma_{ee}| > 1.0$ for all h, unconditionally unstable
- Wave equation: 1^{st} order backward differencing, $\lambda_b h = -\frac{ah}{\Delta x}(1 - cos(k\Delta x) + isin(k\Delta x))$
  - $|\sigma_{ee}| \leq 1.0$ for all some h, conditionally stable
  - Note: $CFL = \frac{ah}{\Delta x}$, CFL Number

- Complex λ-plane, Euler explcit, $\sigma_{ee} = 1 + \lambda h$
  - Let $\lambda h = x + iy$, then $\sigma_{ee} = 1 + x + iy$

$$|\sigma_{ee}| = \sqrt{(1+x)^2 + y^2}$$

  - Contour of $|\sigma_{ee}| = 0.8$ leads to $(1+x)^2 + y^2 = (0.8)^2$: circle in x, y centered at $x = -1$ with radius 0.8 , **Stable**
  - Contour of $|\sigma_{ee}| = 1.2$ leads to $(1+x)^2 + y^2 = (1.2)^{2}$': circle in x, y centered at $x = -1$ with radius 1.2, **Un-Stable**

Example for Euler Implicit

- Euler implicit: $\sigma_{ei} = \frac{1}{1-\lambda h}$
  - Wave equation: central differencing, $\lambda_c = -ai\frac{sin(k\Delta x)}{\Delta x}$

$$\sigma_{ei} = \frac{1}{1 + \frac{ah}{\Delta x} isin(k\Delta x)}$$

  - $|\sigma_{ei}| < 1.0$ for all h, unconditionally stable
  - Even for Compex λh: unconditional stability

- Complex λ-plane, Euler Implcit, $\sigma_{ei} = \frac{1}{1-\lambda h}$
  - Let $\lambda h = x + iy$, then $\sigma_{ei} = \frac{1}{1-x-iy}$

$$|\sigma_{ei}| = \frac{1}{\sqrt{(1-x)^2 + y^2}}$$

  - Contour of $|\sigma_{ei}| = 0.8$ leads to $(1-x)^2 + y^2 = (\frac{1}{0.8})^2$: circle in x, y centered at $x = 1$ with radius $\frac{1}{0.8}$, **Stable**
  - Contour of $|\sigma_{ei}| = 1.2$ leads to $(1-x)^2 + y^2 = (\frac{1}{1.2})^2$: circle in x, y centered at $x = 1$ with radius $\frac{1}{1.2} < 1.0$, **Un-Stable**
  - The unstable contours are in the right half of the inherent stable of the ODE's

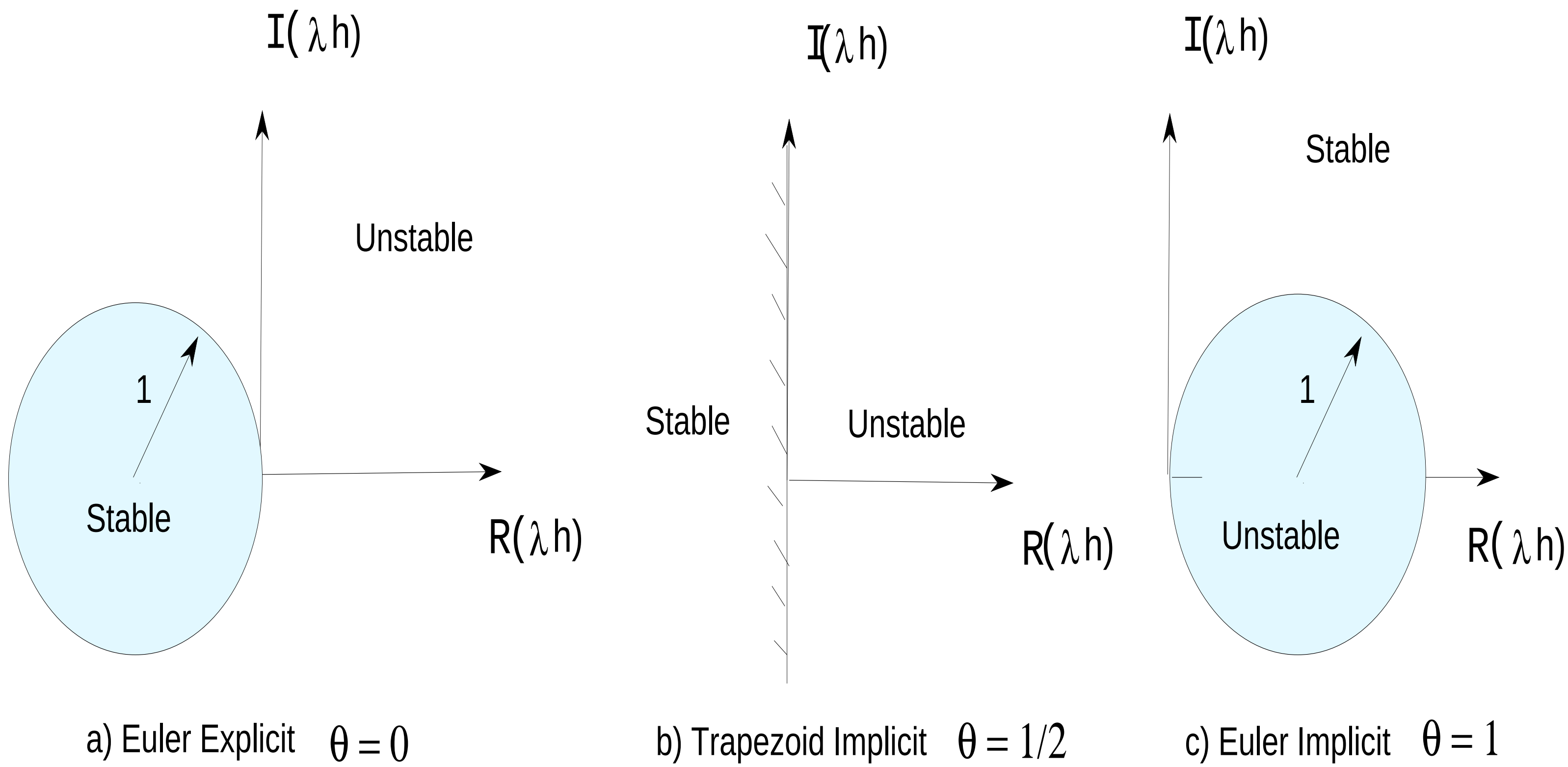
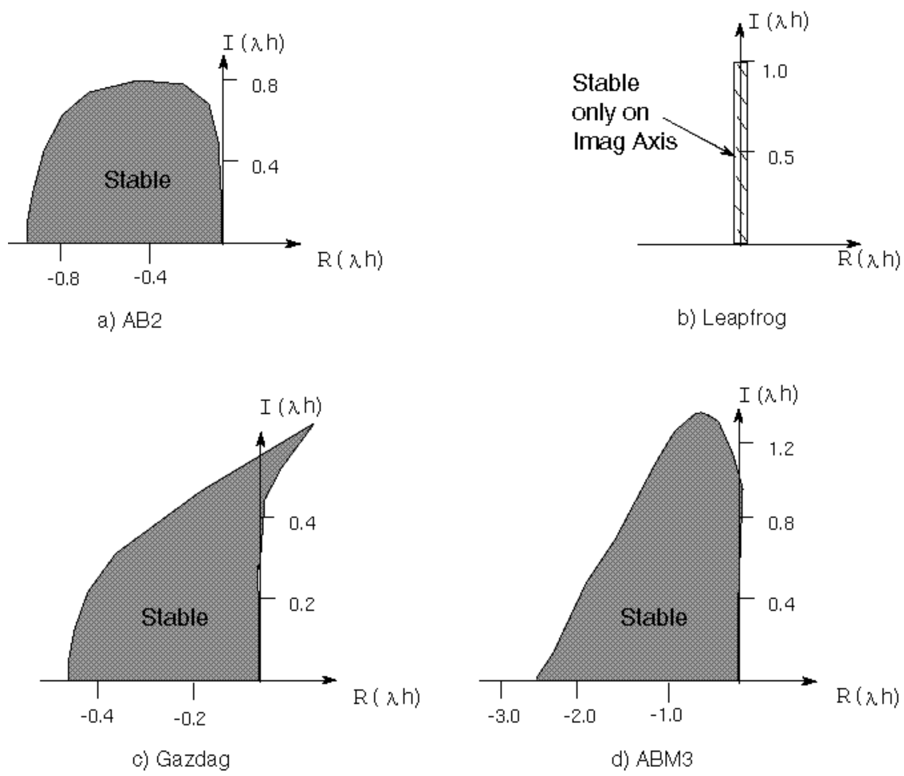
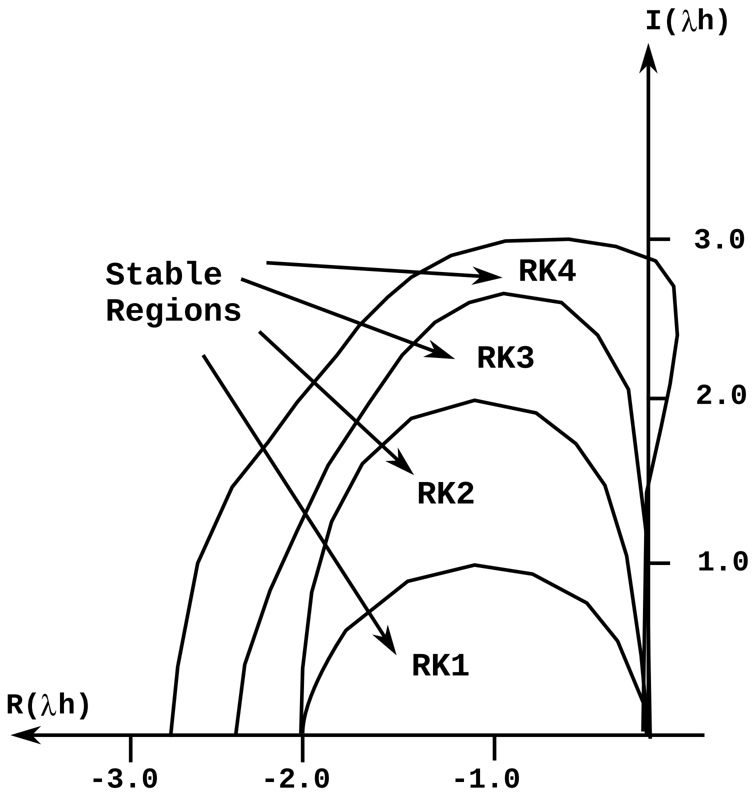


Figure 2: Stability contours for the θ-method.

Stability contours for some explicit methods.



Stability contours for Runge-Kutta methods.



Fourier Stability Analysis

- Classical stability analysis for numerical schemes
- Fourier or von Neumann approach.
  - Periodic in space derivative, similar to modified wave number
  - Usually carried out on point operators
  - Does not depend on an intermediate stage of ODE's.
- *Strictly speaking* it applies only to difference approximations of PDE's that produce OΔE's
- Serves as a fairly reliable *necessary* stability condition, but it is by no means a *sufficient* one.

The Basic Procedure

- Impose a spatial harmonic as an initial value on the mesh
- Will its amplitude grow or decay in time?
- Determined by finding the conditions under which

$$u(x,t) = e^{\alpha t} \cdot e^{i\kappa x} \tag{11}$$

- Is a solution to the *difference* equation, where κ is real and $\kappa\Delta x$ lies in the range $0 \le \kappa\Delta x \le \pi$.
- For the general term,

$$u_{j+m}^{(n+\ell)} = e^{\alpha(t+\ell\Delta t)} \cdot e^{i\kappa(x+m\Delta x)} = e^{\alpha\ell\Delta t} \cdot e^{i\kappa m\Delta x} \cdot u_j^{(n)}$$

- $u_j^{(n)}$ is common to every term and can be factored out.

- Find the term $e^{\alpha \Delta t}$, which we represent by σ, thus:

$$\sigma \equiv e^{\alpha \Delta t}$$

- Since $e^{\alpha t} = \left(e^{\alpha \Delta t}\right)^n = \sigma^n$

$$\boxed{\text{For numerical stability } |\sigma| \leq 1} \tag{12}$$

- Solve for the σ's produced by any given method
- A necessary condition for stability, make sure that, in the worst possible combination of parameters, condition 12 is satisfied.

Example 1

- Finite-difference approximation to the model diffusion equation
- Richardson's method of overlapping steps.

$$u_j^{(n+1)} = u_j^{(n-1)} + \nu \frac{2\Delta t}{\Delta x^2}\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}\right) \tag{13}$$

  - Substitution of Eq. 11 into Eq. 13

$$\sigma = \sigma^{-1} + \nu \frac{2\Delta t}{\Delta x^2}\left(e^{i\kappa\Delta x} - 2 + e^{-i\kappa\Delta x}\right)$$

or

$$\sigma^2 + \underbrace{\left[\frac{4\nu\Delta t}{\Delta x^2}(1 - \cos\kappa\Delta x)\right]}_{2b}\sigma - 1 = 0 \tag{14}$$

  - Eq. 11 is a solution of Eq. 13 if σ is a root of Eq. 14.

– The two roots of Eq. 14 are

$$\sigma_{1,2} = -b \pm \sqrt{b^2 + 1}$$

– One $|\sigma|$ is always > 1.

– Therefore, that by the Fourier stability test, Richardson's method of overlapping steps is unstable for all ν, κ and Δt.

Example 2

- Finite-difference approximation for the model biconvection equation

$$u_j^{(n+1)} = u_j^{(n)} - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right) \tag{15}$$

$$\sigma = 1 - \frac{a\Delta t}{\Delta x} \cdot i \cdot \sin \kappa\Delta x$$

- $|\sigma| > 1$ for all nonzero a and κ.
- Thus we have another finite-difference approximation that, by the Fourier stability test, is unstable for any choice of the free parameters.